JOURNAL OF APPROXIMATION THEORY **10**, 123–130 (1974)

On Interpolation and Approximation by Polynomials with Monotone Derivatives

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Communicated by Oved Shisha

1. INTRODUCTION

A number of results [3–6] in approximation theory have grown out of Shisha's extension [8] of the Weierstrass approximation theorem. Rubinstein [7] phrases one of Shisha's results in the following form: If f is a real function such that, for some integers k, p $(1 \leqslant k \leqslant p)$,

$$f^{(k)}(x) \geqslant 0 \qquad \text{and} \qquad |f^{(p)}(x)| \leqslant M, \qquad 0 \leqslant x \leqslant 1;$$

then, for every integer n $(\geqslant p)$, there exists a polynomial Q_n of degree n (or less) such that

$$Q_n^{(k)}(x) \geqslant 0$$

and

$$|f(x) - Q_n(x)| \leqslant Cn^{k-p}\omega(f^{(p)}, n^{-1}) \tag{1.1}$$

on $[0, 1]$ where $\omega(f^{(p)}, \cdot)$ is the modulus of continuity of $f^{(p)}$ there, and C depends only on p and k.

Rubinstein's paper is based on the following result: If $0 = x_0 < x_1 < \cdots < x_n \leqslant 1$ and $0 = y_0 < y_1 < \cdots < y_n$ are given; then there is a polynomial Q such that $Q'(x) \geqslant 0$ on $(-\infty, \infty)$ while $Q(x_i) = y_i$ for $i = 0, 1, \ldots, n$.

If we view Shisha's quoted result as an existence theorem for a uniform polynomial approximation subject to the constraint of monotonicity, then

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Rubinstein's basic result is an existence theorem for a polynomial interpolation of tabular data subject to a similar constraint. The present paper contains a generalization of Rubinstein's result in a form which emphasizes such interpolation. Moreover, this generalization yields a monotone first-order derivative whenever monotone tabular divided differences warrant such monotonicity. This monotonicity is important in the interpolation of measured data if, for example, physical considerations require positive curvature.

Both Rubinstein's work and our generalization are constructive in the sense that actual interpolation of measured data can be directly developed from the proofs. Rubinstein's monotone interpolation is based on polynomial approximation of positive linear combinations of Heaviside step-functions while our approach is based on a proof due to Kammerer [2].

Section 3 is devoted to monotone approximation and interpolation of k times differentiable functions by polynomials. Here the main theorem is concerned with the degree of such approximation.

2. Interpolation of Convex Data

Let $x_0 < x_1 < \cdots < x_m$ and $y_0 < y_1 < \cdots < y_m$ be given so that the broken line formed by joining (x_{i-1}, y_{i-1}) to (x_i, y_i) for $i = 1, 2, \ldots, m$ is strictly increasing and strictly convex in the sense that the slopes of the successive line segments increase strictly from left to right. The following theorem and lemma generalize the work of Rubinstein [7] to the case of monotone convex data. The proofs are based on a proof due to Kammerer [2].

Lemma 1. *There is a polynomial S satisfying*

$$S(x_i) = y_i, \qquad i = 0, 1, 2, \ldots, m,$$

$$S'(x) \geqslant 0 \qquad \text{on } [x_0, x_m],$$

$$S''(x) \geqslant 0 \qquad \text{on } [x_0, x_m].$$

Proof. It is obvious that for $\epsilon > 0$ and sufficiently small the set of 2^{m+1} broken line functions

$$\mathscr{F} = \{f: f(x_i) = y_i + \epsilon \text{ or } y_i - \epsilon\}$$

will consist entirely of strictly increasing strictly convex broken lines as above. We enumerate the functions in $\mathscr{F}$ and denote the jth function in $\mathscr{F}$ by f_j.

Without loss of generality we may assume $x_0 = 0$ and $x_m = 1$. For each

$f_j \in \mathscr{J}$ and each k the kth Bernstein polynomial $B_k(f_j, \cdot)$ is increasing and convex on $[0, 1]$. Moreover, there is an $N > 0$ so that

$$\max_{0 \leqslant x \leqslant 1} |f_j(x) - B_N(f_j, x)| < \epsilon/2 \qquad \text{for} \quad j = 1, 2, \ldots, \gamma$$

where $\gamma = 2^{m+1}$.

But now $(y_0, y_1, \ldots, y_m)$ is in the convex hull of the set of points

$$\{(B_N(f_j, x_0), B_N(f_j, x_1), \ldots, B_N(f_j, x_m)) : f_j \in \mathscr{J}\}.$$

Thus, there are constants $\alpha_1, \ldots, \alpha_\gamma$ such that each $\alpha_j \geqslant 0$, $\alpha_1 + \alpha_2 + \cdots + \alpha_\gamma = 1$ and

$$y_i = \sum_{j=1}^{\gamma} \alpha_j B_N(f_j, x_i)$$

for $i = 0, 1, \ldots, m$.

So if we set

$$S(x) = \sum_{j=1}^{\gamma} \alpha_j B_N(f_j, x),$$

we have the desired result.

THEOREM 1. (A) *There is a polynomial P satisfying*

$$P(x_i) = y_i, \qquad i = 0, 1, \ldots, m, \tag{2.1}$$

$$P'(x) \geqslant 0 \qquad on\ (-\infty, \infty), \tag{2.2}$$

$$P''(x) \geqslant 0 \qquad on\ [x_0, x_m]. \tag{2.3}$$

(B) *There is a polynomial Q satisfying*

$$Q(x_i) = y_i, \qquad i = 0, 1, \ldots, m, \tag{2.4}$$

$$Q'(x) \geqslant 0 \qquad on\ [x_0, x_m], \tag{2.5}$$

$$Q''(x) \geqslant 0 \qquad on\ (-\infty, \infty). \tag{2.6}$$

Proof. We will prove only part (A) since the proof of (B) is similar. Assume without loss of generality that $0 \leqslant x_0$.

We first show that for any $\epsilon > 0$ there is a polynomial T satisfying (2.2) and (2.3) and

$$|y_i - T(x_i)| < \epsilon \qquad i = 0, 1, \ldots, m.$$

Let $\epsilon > 0$ be given.

Apply Lemma 1 to the broken line connecting the nodes to obtain polynomial $P(x)$ satisfying

$$P(x_i) = y_i, \qquad i = 0, 1,..., m,$$
$$P'(x) \geqslant 0 \qquad \text{on } [x_0, x_m],$$
$$P''(x) \geqslant 0 \qquad \text{on } [x_0, x_m].$$

Now define $H(x) = (P'(x))^{1/2}$ for $x \in [x_0, x_m]$. Then H is increasing on $[x_0, x_m]$ since P' is. Let $1 > \epsilon_1 > 0$ be given. Let Q be a polynomial which satisfies

$$| H(x) - Q(x)| < \epsilon_1 \qquad \text{on } [x_0, x_m]$$

and

$$Q(x) \geqslant 0 \qquad \text{and} \qquad Q'(x) \geqslant 0 \qquad \text{on } [x_0, x_m].$$

This is possible using Bernstein polynomials. Then we have

$$\begin{aligned} | P'(x) - Q^2(x)| &= | H^2(x) - Q^2(x)| \\ &= |(H(x) + Q(x))(H(x) - Q(x))| \\ &\leqslant | H(x) + Q(x)| \, \epsilon_1 \\ &\leqslant (2 \, | H(x)| + \epsilon_1) \, \epsilon_1, \qquad \text{for} \quad x \in [x_0, x_m]. \end{aligned}$$

Thus, by the mean value theorem we have

$$\begin{aligned} &\left| P(x) - P(x_0) - \int_{x_0}^{x} Q^2(t)\, dt \right| \\ &\qquad \leqslant | x - x_0 | \max_{x_0 \leqslant x \leqslant x_m} | P'(x) - Q^2(x)| \\ &\qquad \leqslant | x - x_0 |(2M + 1) \, \epsilon_1, \end{aligned}$$

where

$$M = \max_{x_0 \leqslant x \leqslant x_m} | H(x)| .$$

So, if we set

$$T(x) = P(x_0) + \int_{x_0}^{x} Q^2(t)\, dt,$$

and let

$$\epsilon_1 = \frac{\epsilon}{2(x_m - x_0)(2M + 1)}$$

we have

$$
\begin{aligned}
|y_i - T(x_i)| &= \left| P(x_i) - P(x_0) - \int_{x_0}^{x_1} Q^2(t)\,dt \right| \\
&\leqslant (x_m - x_0)(2M+1)\,\epsilon_1 \\
&< \epsilon, \qquad \text{for} \quad i = 0, 1,\ldots, m, \\
T'(x) &= Q^2(x) \geqslant 0 \text{ on } (-\infty, \infty)
\end{aligned}
$$

and

$$T''(x) = 2Q(x)\,Q'(x) \geqslant 0 \qquad \text{on} \quad [x_0, x_m].$$

The proof of Theorem 1 is now the same as the proof of Lemma 1 if we replace the B_N's by the corresponding T's.

The same methods apply if the data is decreasing and convex (or under similar combinations) to give comparable theorems.

Although Whitmore [9] has successfully applied this result on an IBM 360 (double precision) computer, his method is somewhat more complicated than the method defined here. Thus, our experience in application of these techniques is limited. Practical application will be developed and presented elsewhere.

3. Approximation and Interpolation

In this section we generalize the work of Lorentz and Zeller [3, 4], Roulier [5, 6], and Shisha [8] to approximation and interpolation by polynomials with monotone kth derivatives. Degrees of approximation are obtained.

We begin with the following lemmas.

Lemma 2. *Let $f \in C^k[a, b]$. Suppose that $a < a' < b' < b$. If for a sequence of algebraic polynomials $\{P_n\}$ (P_n of degree n or less) the condition*

$$\max_{a \leqslant x \leqslant b} |f(x) - P_n(x)| = o(n^{-k}) \text{ is satisfied,}$$

then

$$\max_{a' \leqslant x \leqslant b'} |f^{(j)}(x) - P_n^{(j)}(x)| = o\,(n^{j-k}), \qquad j = 1, 2,\ldots, k.$$

The proof of this is found in Roulier [5] and is a modification of a theorem of Garkavi [1].

Lemma 3. *Let $f \in C[a, b]$ and suppose there is a sequence of algebraic* *polynomials* $\{P_n\}$ (P_n *of degree n or less) and a sequence of positive numbers* $\{\epsilon_n\}$ *satisfying*

$$\max_{a\leqslant x\leqslant b} |f(x) - P_n(x)| \leqslant \epsilon_n .$$

Let $m + 1$ *points* $a < x_0 < x_1 < \cdots < x_m < b$ *be given. Then there is a constant C independent of f and n, and a sequence of polynomials* $\{Q_n\}_{n=m}^{\infty}$ (Q_n *of degree n or less) for which*

$$\max_{a\leqslant x\leqslant b} |f(x) - Q_n(x)| \leqslant C\epsilon_n$$

and

$$Q_n(x_i) = f(x_i), \qquad i = 0, 1,..., m.$$

Proof. Suppose $n \geqslant m$ and $|f(x) - P_n(x)| \leqslant \epsilon_n$ for all x in $[a, b]$. Let $\delta_i = f(x_i) - P_n(x_i)$, $i = 0,..., m$. Let R_m be Lagrange's interpolating polynomial of degree $\leqslant m$ satisfying $R_m(x_i) = \delta_i$. It is easy to see that there is a K independent of f and n for which

$$| R_m(x)| \leqslant K\epsilon_n .$$

Then set $Q_n(x) = P_n(x) + R_m(x)$, and we have

$$|f(x) - Q_n(x)| \leqslant |f(x) - P_n(x)| + | R_m(x)| \leqslant \epsilon_n + K\epsilon_n = (1 + K)\epsilon_n .$$

LEMMA 4. *Let* $f \in C^k[a, b]$, *and let* $\omega(f^{(k)}, \cdot)$ *be the modulus of continuity of* $f^{(k)}$ *on* $[a, b]$. f *may be extended to a function* $F \in C^k[a - 1, b + 1]$ *in such a way that the modulus of continuity* $\omega(F^{(k)}, \cdot)$ *satisfies*

$$\omega(F^{(k)}, h) \leqslant \omega(f^{(k)}, h) \qquad \text{for} \quad h \leqslant b - a.$$

The proof of this is in Roulier [5] and will not be repeated here.

We now give the main theorem of this section.

THEOREM 2. *Let* $k_1 < k_2 < \cdots < k_p$ *be fixed positive integers and let* $\epsilon_1 ,..., \epsilon_p$ *be fixed signs (i.e.,* $\epsilon_j = \pm 1$). *Suppose* $f \in C^k[a, b]$ *and* $k_p \leqslant k$. *Assume*

$$\epsilon_i f^{(k_i)}(x) > 0 \qquad \text{for} \quad a \leqslant x \leqslant b \qquad \text{and} \quad i = 1, 2,..., p. \tag{3.1}$$

Suppose $m + 1$ *points are given so that*

$$a \leqslant x_0 < x_1 < \cdots < x_m \leqslant b.$$

Then for n sufficiently large there are polynomials P_n of degree less than or equal to n for which

$$\epsilon_j P_n^{(k_j)}(x) > 0 \quad \text{on} \quad [a, b], \quad j = 1, 2, \ldots, p, \tag{3.2}$$

$$P_n(x_i) = f(x_i), \quad i = 0, 1, \ldots, m, \tag{3.3}$$

$$\max_{a \leqslant x \leqslant b} | f(x) - P_n(x)| \leqslant (C/n^k)\, \omega(1/n), \tag{3.4}$$

where C is a constant depending only on $x_0, \ldots, x_m$ and ω is the modulus of continuity of $f^{(k)}$ on $[a, b]$.

Proof. Extend f to a function $F \in C^k[a - 1, b + 1]$ as in Lemma 4. For each n let Q_n be the polynomial of best approximation to F on $[a - 1, b + 1]$. By Jackson's theorem we know that for some constant K

$$\max_{a-1 \leqslant x \leqslant b+1} | F(x) - Q_n(x)| \leqslant (K/n^k)\, \omega(1/n).$$

By Lemma 3 there is a sequence of polynomials $\{R_n\}_{n=m}^{\infty}$ such that

$$\max_{a-1 \leqslant x \leqslant b+1} | F(x) - R_n(x)| \leqslant (C/n^k)\, \omega(1/n)$$

and

$$R_n(x_i) = f(x_i), \quad i = 0, 1, \ldots, m.$$

By Lemma 2 we see that

$$\max_{a \leqslant x \leqslant b} | f(x) - R_n^{(j)}(x)| = o(1/n^{k-j})$$

for $j = 0, 1, \ldots, k$.

Thus, in particular, we have $R_n^{(k_i)}(x) \to f^{(k_i)}(x)$ uniformly on $[a, b]$ for $i = 1, \ldots, p$. This together with the strictness of (3.1) completes the proof. ∎

References

1. A. L. Garkavi, Simultaneous approximation of periodic functions and their derivatives by trigonometric polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* **24** (1960), 103–128.
2. W. J. Kammerer, Polynomial approximations to finitely oscillating functions, *Math. Comp.* **15** (1961), 115–119.
3. G. G. Lorentz and K. L. Zeller, Degree of approximation by monotone polynomials I, *J. Approximation Theory* **1** (1968), 501–504.
4. G. G. Lorentz and K. L. Zeller, Degree of approximation by monotone polynomials II, *J. Approximation Theory* **2** (1969), 265–269.
5. J. A. Roulier, Monotone and weighted approximation, Doctoral Dissertation, Syracuse University, 1968.

6. J. A. Roulier, Monotone approximation of certain classes of functions, *J. Approximation Theory* **1** (1968), 319–324.
7. Z. Rubinstein, On polynomial δ-type functions and approximation by monotonic polynomials, *J. Approximation Theory* **3** (1970), 1–6.
8. O. Shisha, Monotone approximation, *Pacific J. Math.* **15** (1965), 667–671.
9. R. W. Whitmore, Monotonic polynomial approximation, M. S. Thesis, Texas Tech. University, 1971.